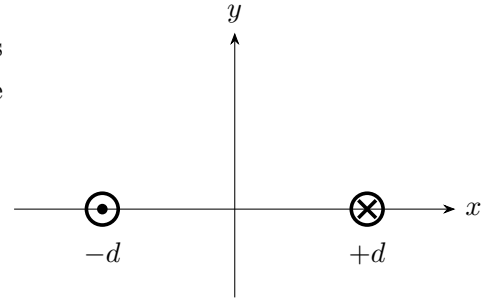


1. Let two long parallel wires, a distance $2d$ apart, carry equal currents I in the opposite direction. One wire is at $(x = -d, y = 0)$ and the other wire is at $(x = d, y = 0)$.

[12 Pts.] (a) Find the magnetic field B on the x-axis.

[13 Pts.] (b) Find the magnetic field B on the y-axis.

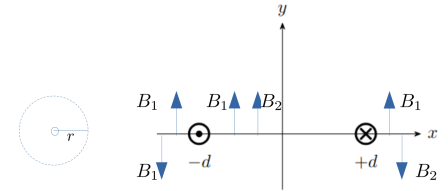


a) Ampere's law $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I \Rightarrow |\vec{B}| = \frac{\mu_0 I}{2\pi r}$

$$-\infty < x < -d: \quad \vec{B} = \frac{\mu_0 I}{2\pi} \left(\frac{1}{|x| + d} - \frac{1}{|x| - d} \right) \hat{j}$$

$$-d < x < d: \quad \vec{B} = \frac{\mu_0 I}{2\pi} \left(\frac{1}{|x| + d} + \frac{1}{d - |x|} \right) \hat{j}$$

$$d < x < \infty: \quad \vec{B} = \frac{\mu_0 I}{2\pi} \left(\frac{1}{x + d} - \frac{1}{x - d} \right) \hat{j}$$

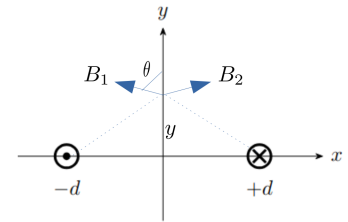


All equal to $\vec{B} = \frac{\mu_0 I}{2\pi} \left(\frac{1}{x+d} - \frac{1}{x-d} \right) \hat{j} = \frac{\mu_0 I}{\pi} \frac{d}{x^2 - d^2} (-\hat{j})$

b) $\vec{B}_T = \vec{B}_1 + \vec{B}_2 = |\vec{B}| \hat{j} = 2|\vec{B}_1| \cos \theta \hat{j}$

where $\cos \theta = \frac{d}{\sqrt{d^2 + y^2}}$

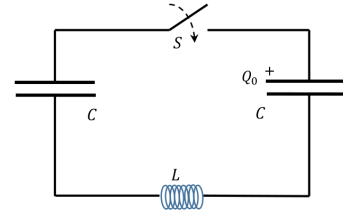
$$|\vec{B}_1| = \frac{\mu_0 I}{2\pi(d^2 + y^2)} \Rightarrow \vec{B}_T(y) = \frac{\mu_0 I}{\pi} \frac{d}{(d^2 + y^2)} \hat{j}$$



Checking $(x, y) = (0, 0): \quad B_T(0) = \frac{\mu_0 I}{\pi d}$

2. Consider the circuit with two capacitors of capacitance C and inductor with inductance L , shown in the figure. Initially, the switch S is open and one of the capacitors is charged with total charge Q_0 and the other one is uncharged. At time $t = 0$ the switch S in the circuit is closed.

- [8 Pts.] (a) What is the oscillation frequency of the current in the circuit?
- [8 Pts.] (b) What is the maximum current in the circuit?
- [9 Pts.] (c) At which time(s) is the energy stored in the inductor maximum?

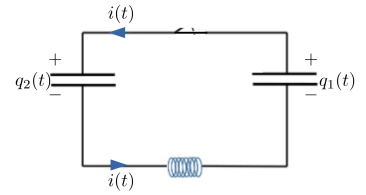


$$a) q_1(t) + q_2(t) = Q_0, \quad V_{C1} = \frac{q_1}{C}, \quad V_{C2} = \frac{q_2}{C} \quad \text{and} \quad i(t) = -\frac{dq_1}{dt} = \frac{dq_2}{dt}$$

$$V_L + V_{C1} - V_{C2} = 0, \quad V_L = L \frac{di}{dt} = L \frac{d^2 q_2}{dt^2}$$

$$L \frac{d^2 q_2}{dt^2} - \frac{q_1}{C} + \frac{q_2}{C} = 0$$

$$\frac{d^2 q_2}{dt^2} = \frac{1}{LC} (Q_0 - 2q_2)$$



$$\text{Let } q = Q_0 - 2q_2$$

$$\frac{d^2 q}{dt^2} + \frac{2}{LC} q = 0$$

$$q = A \cos(\omega t) + B \sin(\omega t) \quad \text{and} \quad \omega = \sqrt{\frac{2}{LC}}$$

$$\text{At } t = 0 \quad q_2 = 0, \text{ Then } A = -Q_0$$

$$i(t) = \frac{dq_2}{dt} = \frac{Q_0 \omega}{2} \sin(\omega t) + \frac{B \omega}{2} \cos(\omega t) \quad , \text{ at } t = 0 \quad i = 0, \text{ then } B = 0$$

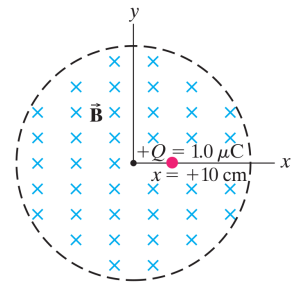
$$\Rightarrow i(t) = \frac{Q_0 \omega}{2} \sin(\omega t) \quad , \text{ and } \quad f = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{2}{LC}} = \frac{1}{\sqrt{2\pi^2 LC}}$$

b)

$$i_{\max} = \frac{Q_0 \omega}{2} = \frac{Q_0}{2} \sqrt{\frac{2}{LC}} = \frac{Q_0}{\sqrt{2LC}}$$

$$c) U_C = \frac{1}{2} L I^2, \quad U_{\max} = \frac{1}{8} L Q_0^2 \omega^2 \quad \text{at} \quad t = \pm \left(n + \frac{1}{2}\right) \frac{\pi}{\omega} = \pm \left(n + \frac{1}{2}\right) \pi \sqrt{\frac{LC}{2}}, \quad n = 0, 1, 2, \dots$$

3. In a circular region with radius a , there is a uniform magnetic field B pointing into the page. An xy coordinate system has its origin at the circular region's center. A free positive point charge $+Q = 1.0 \mu\text{C}$ is initially at rest at a position $x = +10 \text{ cm}$ on the x axis.



- [13 Pts.] (a) If the magnitude of the magnetic field is now decreased at a rate of -0.10 T/s , what force will act on $+Q$ (magnitude and direction)?
- [12 Pts.] (b) If the magnetic field magnitude is given as $B(t) = B_0 - \alpha t$, what is the magnitude of the Poynting vector at point $(0, a)$ at time t . Draw the Poynting vector at that point. (Ignore the electric field created by the point charge Q .)

a)

$$\oint \vec{E} \cdot d\vec{\ell} = E(2\pi r) = -\frac{d\Phi_B}{dt} = -(\pi r^2) \frac{dB}{dt} \quad \Rightarrow \quad E = -\frac{r}{2} \frac{dB}{dt}$$

$$\vec{F} = Q\vec{E} = -\frac{Qr}{2} \frac{dB}{dt} = F = 5 \text{ } (-\hat{j}) \text{ nN}$$

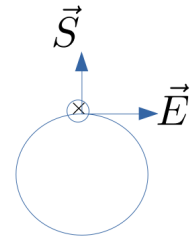
Induced circular current flows in the clockwise direction, so force is in the $-\hat{j}$ direction.

b)

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}, \quad E = -\frac{a}{2} \frac{dB}{dt} = \frac{a\alpha}{2}$$

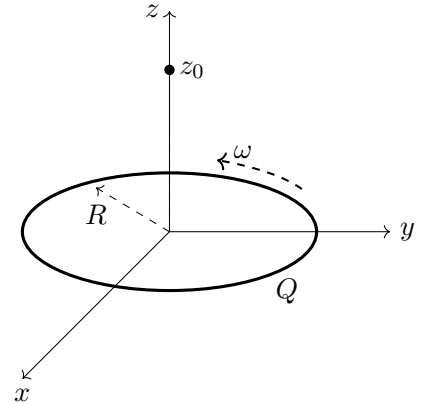
$$\vec{S} = \frac{1}{\mu_0} \left(\frac{a\alpha}{2} \hat{i} \times (B_0 - \alpha t)(-\hat{k}) \right) = \frac{a\alpha}{2\mu_0} (B_0 - \alpha t)(\hat{i} \times -\hat{k})$$

$$\vec{S} = \frac{a\alpha}{2\mu_0} (B_0 - \alpha t) \hat{j}$$



4. A thin insulating ring of radius R lies in the xy -plane with its center at the origin. The ring carries a total charge Q , distributed uniformly along its circumference.

- [8 Pts.] (a) Find the electric field at a point located on the z -axis at height z_0 above the center of the ring.
- [8 Pts.] (b) Suppose the ring rotates about the z -axis with constant angular velocity ω . Determine the net electric current produced by this rotation.
- [9 Pts.] (c) Find the magnitude of the magnetic field at the same point on the z -axis as in part (a).



a)

$$\begin{aligned} \text{(a)} \quad V(z) &= \frac{Q}{4\pi\epsilon_0} \frac{1}{\sqrt{z^2 + R^2}} \quad \vec{E}(0, 0, z) = E(z)\hat{k}, \\ E(z) &= -\frac{dV}{dz} = -\frac{Q}{4\pi\epsilon_0} \frac{-z}{(z^2 + R^2)^{3/2}} \\ \vec{E}(z_0) &= \frac{Qz_0}{4\pi\epsilon_0(z_0^2 + R^2)^{3/2}} \hat{k} \end{aligned}$$

b) One period is

$$T = \frac{2\pi}{\omega}, \quad I = \frac{Q}{T} = \frac{Q\omega}{2\pi}$$

c) $\vec{B}(0, 0, z) = B(z)\hat{k}$ by symmetry

$$\begin{aligned} |d\vec{B}| &= \frac{\mu_0 I}{4\pi} \frac{|d\vec{l}|}{(R^2 + z^2)}, \quad |d\vec{l}| = R d\phi \quad \text{and} \quad dB_z = |d\vec{B}| \sin \theta = |d\vec{B}| \frac{R}{\sqrt{R^2 + z^2}} \\ \Rightarrow dB_z &= \frac{\mu_0 I}{4\pi} \frac{R}{(R^2 + z^2)^{3/2}} R d\phi \end{aligned}$$

$$\begin{aligned} B_z &= \frac{\mu_0 I}{4\pi} \frac{R^2}{(R^2 + z^2)^{3/2}} \int_0^{2\pi} d\phi = \frac{\mu_0 I R^2}{2(R^2 + z^2)^{3/2}} \\ \Rightarrow \vec{B}(z_0) &= \frac{\mu_0 Q \omega R^2}{4\pi(R^2 + z_0^2)^{3/2}} \hat{k} \end{aligned}$$

